JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **35**, No. 3, August 2022
http://dx.doi.org/10.14403/jcms.2022.35.3.227

NOTE ON THE PINNED DISTANCE PROBLEM OVER FINITE FIELDS

Doowon Koh

Abstract. Let $\mathbb{F}_q$ be a finite field with odd q elements. In this article, we prove that if $E \subseteq \mathbb{F}_q^d, d \geq 2$, and $|E| \geq q$, then there exists a set $Y \subseteq \mathbb{F}_q^d$ with $|Y| \sim q^d$ such that for all $y \in Y$, the number of distances between the point y and the set E is $\sim q$. As a corollary, we obtain that for each set $E \subseteq \mathbb{F}_q^d$ with $|E| \geq q$, there exists a set $Y \subseteq \mathbb{F}_q^d$ with $|Y| \sim q^d$ so that any set $E \cup \{y\}$ with $y \in Y$ determines a positive proportion of all possible distances. The averaging argument and the pigeonhole principle play a crucial role in proving our results.

1. Introduction

Let $\mathbb{F}_q^d$ be the d-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements. In 2005, Iosevich and Rudnev [5] initially posed and studied an analogue of the Falconer distance problem over finite fields. They asked for the minimal exponent $\alpha > 0$ such that if $E \subseteq \mathbb{F}_q^d$ and $|E| \geq Cq^\alpha$ for a sufficiently large constant $C > 0$, then

$$|\Delta(E)| \geq cq$$

for some constant $0 < c \leq 1$, where $|\Delta(E)|$ denotes the cardinality of the distance set $\Delta(E)$, defined by

$$\Delta(E) = \{||x - y|| : x, y \in E\}.$$

Here we recall that $||\alpha|| := \sum_{j=1}^{d} \alpha_j^2$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{F}_q^d$.

Received June 8, 2022; Accepted August 24, 2022.
2010 Mathematics Subject Classification: Primary 52C10 ; Secondary 11T23 .
Key words and phrases: Finite field, Pinned distance.
This work was conducted during the research year of Chungbuk National University in 2022, and was supported by Basic Science Research Programs through National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B07044469).

By developing the discrete Fourier machinery, Iosevich and Rudnev [5] proved that $|\Delta(E)| \sim q$ whenever $|E| \geq Cq^{(d+1)/2}$. We recall that $A \ll B$ means that $A \leq CB$ for some constant $C > 0$, which is independent of q, and we use $A \sim B$ if $A \ll B$ and $B \ll A$. The authors in [4] showed that the exponent $(d+1)/2$ is optimal for all odd dimensions $d \geq 3$ except for the cases when -1 is not a square and $d = 4k-1$ for $k \in \mathbb{N}$. However, in any other cases including even dimensions $d \geq 2$, it has been conjectured by Iosevich and Rudnev [5] that in order to have a positive proportion of all distances, the exponent $(d+1)/2$ can be improved to $d/2$.

Conjecture 1.1 (Iosevich-Rudnev's Conjecture). *Let $E \subseteq \mathbb{F}_q^d$. Suppose that $d \geq 2$ is even or $d, q \equiv 3 \mod 4$. Then if $|E| \geq Cq^{d/2}$ for a sufficiently large constant $C > 0$, we have $|\Delta(E)| \sim q$.*

Iosevich-Rudnev's Conjecture is still open and even the threshold $(d+1)/2$ has not been improved except for two dimensions. In the case of $d = 2$ over general finite fields, the authors in [2] obtained the $4/3$ exponent, which is the first result to break down the exponent $(d+1)/2$. This result was obtained by applying the sharp restriction estimates for the circles on the plane. More precisely, they proved the following result with an explicit constant.

Theorem 1.2 ([2]). *Let E be a subset of $\subset \mathbb{F}_q^2$ with $|E| \geq q^{4/3}$. Then following statements hold:*

- *If $q \equiv 3 \mod 4$, then $|\Delta(E)| \geq \frac{q}{1+\sqrt{3}}$.*
- *If $q \equiv 1 \mod 4$, then $|\Delta(E)| \geq C_q q$, where the constant C_q is defined by*
$$C_q := \frac{\left(1-2q^{-1}\right)^2}{1+\sqrt{3}-\sqrt{3}q^{-2/3}}.$$

Notice that $C_q > 0$ for all $q \geq 3$, and C_q converges to $\frac{1}{1+\sqrt{3}}$ as $q \to \infty$. Therefore, we see that there is a constant $c > 0$, independent of q, such that $C_q \geq c > 0$. From this observation, the following corollary is a direct consequence of Theorem 1.2.

Corollary 1.3 ([2]). *Suppose that $E \subseteq \mathbb{F}_q^2$ with $|E| \geq q^{4/3}$. Then we have*
$$|\Delta(E)| \sim q.$$

Using a group action approach, Bennett, Hart, Iosevich, Pakianathan, and Rudnev [1] provided an alternative proof of the exponent $4/3$ in the above corollary.

As a strong version of the Falconer distance problem, one has studied the pinned distance problem over finite fields. Given $E \subseteq \mathbb{F}_q^d, d \geq 2$, and $y \in \mathbb{F}_q^d$, the pinned distance set with a pin y, denoted by $\Delta_y(E)$, is defined by

$$\Delta_y(E) = \{||x-y|| : x \in E\}.$$

The Chapman, Erdoğan, Hart, Iosevich, and Koh [2] showed that the exponent $(d+1)/2$ due to Iosevich and Rudnev holds true for the pinned distance sets. More precisely they proved the following.

THEOREM 1.4 ([2]). *Let $E \subseteq \mathbb{F}_q^d, d \geq 2$. If $|E| \geq q^{\frac{d+1}{2}}$, then there exists a subset E' of E with $|E'| \sim |E|$ so that for every $y \in E'$, we have*

$$|\Delta_y(E)| \sim q.$$

As seen in the conjecture of the Falconer distance set problem, the exponent $(d+1)/2$ cannot be improved except for the cases when $d, q \equiv 3 \mod 4$ or $d \geq 2$ is even. However, in those cases it have been believed that $d/2$ can be the best possible exponent for the pinned distance sets. As partial evidence for this prediction, the $4/3$ exponent result was extended to the pinned distance sets in $\mathbb{F}_q^2$ by Hanson, Lund, and Roche-Newton [3], who successfully performed the bisector energy estimate.

THEOREM 1.5 ([3]). *Let $E \subseteq \mathbb{F}_q^2$. If $|E| \geq q^{4/3}$, then the conclusion of Theorem 1.4 holds.*

When q is prime, the exponent $4/3$ has been improved to $5/4$ by Murphy, Petridis, Pham, Rudnev, and Stevenson [6].

THEOREM 1.6 ([6]). *Let q be prime. Then if $E \subseteq \mathbb{F}_q^2$ with $|E| \geq q^{\frac{5}{4}}$, we have*

$$\max_{y \in E} |\Delta_y(E)| \sim q.$$

Despite researchers' efforts, the conjectured exponent $d/2$ has not been proven. This problem is unlikely to be solved with currently known techniques. Moreover, there are only few evidences to support that the conjecture is true.

The main purpose of this paper is not to derive an improved result on the distance problem but to address that the possibility that random sets satisfy the distance conjecture is very high.

1.1. The statement of main results

Our main theorem is as follows.

THEOREM 1.7. *Suppose that $E \subseteq \mathbb{F}_q^d, d \geq 2$, with $|E| \geq q$. Then, for any $a > 1$, there exists $Y \subseteq \mathbb{F}_q^d$ with $|Y| \geq \frac{a-1}{a} q^d$ so that for all $y \in Y$, we have*

$$|\Delta_y(E)| \geq \frac{q}{2a}.$$

In the above theorem, the set Y depends on the set E and $a > 0$. Let us fix a constant $a > 0$, independent of q. Then the following statements can be considered.

- If $E \cap Y \neq \phi$ and $|E| = q$, then the set E yields at least $\sim q$ pinned distances.
- On the other hand, if $E \cap Y = \phi$ and $|E| = q$, then Theorem 1.7 does not provide any information about the size of the pinned distance set determined by the set E.

Here, a problem is that given a set $E \subset \mathbb{F}_q^d$ with $|E| = q$, we do not know if $E \cap Y = \phi$ or not. However, for each $y \in \mathbb{F}_q^d$, we can expect with at least $100(a-1)/a\%$ certainty that the set $E \cup \{y\}$ determines at least $\frac{q}{2a}$ distinct pinned distances with the pin y. For example, if we take $a = 100$, then with at least 99% certainty we may claim that $E \cap \{y\}$ for any $y \in \mathbb{F}_q^d$ generates at least $q/200$ distinct pinned distances.

2. Proof of main result (Theorem 1.7)

To complete the proof, we will prove the following proposition which clearly implies Theorem 1.7

PROPOSITION 2.1. *Let $E \subseteq \mathbb{F}_q^d$. Then given $a > 1$, there exists $Y \subseteq \mathbb{F}_q^d$ with $|Y| \geq \frac{a-1}{a} q^d$ such that for all $y \in Y$,*

$$|\Delta_y(E)| \geq \min\left\{\frac{q}{2a}, \frac{|E|}{2a}\right\}.$$

To prove this proposition, we begin with the standard counting argument as in [2].

To find a lower bound of the cardinality of the y-pinned distance set $\Delta_y(E)$, we consider the y-pinned counting function $\nu_y : \mathbb{F}_q \to \mathbb{N} \cup \{0\}$, which maps an element t in $\mathbb{F}_q$ to the number of elements x in E such that $||x-y|| = t$. In other words, for $y \in \mathbb{F}_q^d, t \in \mathbb{F}_q$, we have

$$\nu_y(t) = \sum_{x \in E: ||x-y||=t} 1.$$

Since $|E|^2 = \left(\sum_{t \in \Delta_y(E)} \nu_y(t)\right)^2$, it follows from the Cauchy-Schwarz inequality that

$$|\Delta_y(E)| \geq \frac{|E|^2}{\sum_{t \in \mathbb{F}_q} \nu_y^2(t)}. \tag{2.1}$$

Thus our problem is reduced to finding a good upper bound of the quantity $\sum_{t \in \mathbb{F}_q} \nu_y^2(t)$, which will be conducted in the following subsection (see Lemma 2.3).

2.1. Key lemmas

The average of $\sum_{t \in \mathbb{F}_q} \nu_y^2(t)$ over y in $\mathbb{F}_q^d$ is explicitly given as follows:

LEMMA 2.2. *Let $E \subseteq \mathbb{F}_q^d$. Then we have*

$$\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) = \frac{|E|^2}{q} + \frac{q-1}{q}|E|.$$

Proof. By the definition of the y-pinned counting function $\nu_y(t)$, we have for each $y \in \mathbb{F}_q$,

$$\sum_{t \in \mathbb{F}_q} \nu_y^2(t) = \sum_{x,z \in E: ||x-y||=||z-y||} 1.$$

Hence, the average of it over $y \in \mathbb{F}_q^d$ is given as follows:

(2.2)
$$\begin{aligned}\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) &= \frac{1}{q^d} \sum_{\substack{x,z \in E\\ :x=z}} \sum_{\substack{y \in \mathbb{F}_q^d\\ :||x-y||=||z-y||}} 1 + \frac{1}{q^d} \sum_{\substack{x,z \in E\\ :x \neq z}} \sum_{\substack{y \in \mathbb{F}_q^d\\ :||x-y||=||z-y||}} 1\\ &= |E| + \frac{1}{q^d} \sum_{\substack{x,z \in E\\ :x \neq z}} \sum_{\substack{y \in \mathbb{F}_q^d\\ :||x-y||=||z-y||}} 1.\end{aligned}$$

Now, we notice that for $x, z \in E$ with $x \neq z$, we have

$$\sum_{y \in \mathbb{F}_q^d: ||x-y||=||z-y||} 1 = q^{d-1}. \tag{2.3}$$

In fact, since $x \neq z$, the quantity $\sum_{y\in\mathbb{F}_q^d:||x-y||=||z-y||} 1$ is the number of the elements in the hyper-plane which bisects the line segment joining x and z. Alternatively we can prove this rigorously by using the finite field Fourier analysis. To see this, let χ denote a nontrivial additive character of $\mathbb{F}_q$. Then, by the orthogonality of χ, we see that if $x \neq z$, then

$$\begin{aligned}\sum_{y\in\mathbb{F}_q^d:||x-y||=||z-y||} 1 &= q^{-1}\sum_{y\in\mathbb{F}_q^d}\sum_{s\in\mathbb{F}_q}\chi(s(||x-y||-||z-y||))\\ &= q^{d-1}+q^{-1}\sum_{y\in\mathbb{F}_q^d}\sum_{s\neq 0}\chi(s(||x-y||-||z-y||)).\end{aligned}$$

Applying the orthogonality of χ to the sum over y, we see that the second term above is zero since the equation

$$\chi(s(||x-y||-||z-y||)) = \chi(-2s(x-z)\cdot y)\chi(s(||x||-||z||))$$

holds and $s(x-z)$ is not a zero vector for $s \neq 0, x \neq z$. Hence, the equation (2.3) holds.

Finally, combining the above two estimates (2.2), (2.3), we obtain the desirable estimate.

□

The following result can be obtained by the pigeonhole principle together with Lemma 2.2.

LEMMA 2.3. *Let $E \subseteq \mathbb{F}_q^d$. Then for any $a > 1$, there exists $Y \subseteq \mathbb{F}_q^d$ with $|Y| \geq \frac{a-1}{a}q^d$ such that for every $y \in Y$,*

$$\sum_{t\in\mathbb{F}_q}\nu_y^2(t) \leq \frac{a}{q}|E|^2 + \frac{a(q-1)}{q}|E|.$$

Proof. Let us fix $a > 1$. Define

$$Y = \left\{ y \in \mathbb{F}_q^d : \sum_{t\in\mathbb{F}_q}\nu_y^2(t) \leq \frac{a}{q}|E|^2 + \frac{a(q-1)}{q}|E| \right\}.$$

To complete the proof, it remains to show that

$$|Y| \geq \frac{a-1}{a}q^d.$$

By contradiction, let us assume that

$$|Y| < \frac{a-1}{a}q^d. \tag{2.4}$$

It is clear that

$$\mathbb{F}_q^d \setminus Y = \left\{ y \in \mathbb{F}_q^d : \sum_{t \in \mathbb{F}_q} \nu_y^2(t) > \frac{a}{q}|E|^2 + \frac{a(q-1)}{q}|E| \right\}. \tag{2.5}$$

We also notice that for all $y \in \mathbb{F}_q^d$,

$$\sum_{t \in \mathbb{F}_q} \nu_y^2(t) \geq \sum_{t \in \mathbb{F}_q} \nu_y(t) = |E|. \tag{2.6}$$

Now by Lemma 2.2, it follows that

$$\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) = \frac{|E|^2}{q} + \frac{q-1}{q}|E|. \tag{2.7}$$

However, we can also estimate it as follows. Using (2.5) and (2.6), we have

$$\begin{aligned}
&\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) \\
&= \frac{1}{q^d} \sum_{y \in Y} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) + \frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d \setminus Y} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) \\
&> \frac{1}{q^d}|Y||E| + \frac{1}{q^d}(q^d - |Y|) \left(\frac{a}{q}|E|^2 + \frac{a(q-1)}{q}|E| \right) \\
&= \frac{a|E|^2}{q} + \frac{a(q-1)|E|}{q} + \left(\frac{|E|}{q^d} - \frac{a|E|^2}{q^{d+1}} - \frac{a(q-1)|E|}{q^{d+1}} \right) |Y|.
\end{aligned}$$

Since $a > 1$, in the third term above, the coefficient of $|Y|$ is negative. Hence, we can combine the above estimate with (2.4) to deduce that

$$\begin{aligned}
\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) > \frac{a|E|^2}{q} &+ \frac{a(q-1)|E|}{q} \\
&+ \left(\frac{|E|}{q^d} - \frac{a|E|^2}{q^{d+1}} - \frac{a(q-1)|E|}{q^{d+1}} \right) \left(\frac{a-1}{a} q^d \right)
\end{aligned}$$

Simplifying the RHS of the above estimate, we get

$$\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) > \frac{|E|^2}{q} + \frac{q-1}{q}|E| + \frac{a-1}{a}|E|,$$

which contradicts the equation (2.7) since $a > 1$. $\square$

2.2. Proof of Proposition 2.1

Combining (2.1) and Lemma 2.3, we get the required result:

$$|\Delta_y(E)| \geq \frac{|E|^2}{\frac{a}{q}|E|^2 + \frac{a(q-1)}{q}|E|} \geq \min\left\{\frac{q}{2a},\ \frac{q|E|}{2a(q-1)}\right\} \geq \min\left\{\frac{q}{2a},\ \frac{|E|}{2a}\right\}.$$

References

[1] M. Bennett, D. Hart, A. Iosevich, J. Pakianathan, and M. Rudnev, *Group actions and geometric combinatorics in* $\mathbb{F}_q^d$, Forum Math., **29** (2017), no. 1, 91-110.

[2] J. Chapman, M. Erdoğan, D. Hart, A. Iosevich, and D. Koh, *Pinned distance sets, Wolff's exponent in finite fields and sum-product estimates*, Math. Z., **271** (2012), no. 1-2, 63-93.

[3] B. Hanson, B. Lund, and O. Roche-Newton, *On distinct perpendicular bisectors and pinned distances in finite fields,* Finite Fields Appl., **37** (2016), 240-264.

[4] D. Hart, A. Iosevich, D. Koh and M. Rudnev, *Averages over hyperplanes, sum-product theory in vector spaces over finite fields and the Erdös-Falconer distance conjecture*, Trans. Amer. Math. Soc., **363** (2011), no. 6, 3255-3275.

[5] A. Iosevich, M. Rudnev, *Erdős distance problem in vector spaces over finite fields*, Trans. Amer. Math. Soc., **359** (2007), no. 12, 6127-6142.

[6] B. Murphy, G. Petridis, T. Pham, M. Rudnev, and S. Stevenson, *On the pinned distances problem in positive characteristic,* J. Lond. Math. Soc., (2), **105** (2022), no. 1, 469-499.

Doowon Koh
Department of Mathematics
Chungbuk National University
Cheongju, Chungbuk 28644, Republic of Korea
E-mail: `koh131@chungbuk.ac.kr`